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Steady Navier–Stokes equations in a domain with piecewise smooth boundary

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Abstract

We are concerned with the boundary value problem for the steady Navier–Stokes equations in a 2D bounded domain with piecewise smooth boundary. Existence and uniqueness of the solution to the above problem is proved in weighted Sobolev spaces by means of the Mellin transform and the regularizer method.

Keywords: Steady Navier–Stokes equations; 2D bounded domain with piecewise smooth boundary; Weighted Sobolev spaces

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ whose boundary consists of two smooth curves Γ and Σ with common endpoints $x^{(1)}$ and $x^{(2)}$. For simplicity we assume that $\Gamma, \Sigma \in C^{l+2}$ with a non-negative integer l, and that $x^{(1)}$ and $x^{(2)}$, respectively, have neighbourhoods $U^{(1)}$ and $U^{(2)}$ in $\mathbb{R}^2$ such that $U^{(1)} \cap \bar{\Omega}$ and $U^{(2)} \cap \bar{\Omega}$ are diffeomorphic to $B(0; \delta^{(1)}) \cap \bar{d}_{\theta^{(1)}}$ and $B(0; \delta^{(2)}) \cap \bar{d}_{\theta^{(2)}}$. Here $B(0; \delta)$ is an open disc with the center at the origin and the radius δ and $d_\theta = \{z = (r\cos\phi, r\sin\phi) \mid r > 0,\ 0 < \phi < \theta\}$ with $0 < \theta < 2\pi$.

In this paper we consider the following boundary value problem for the steady Navier–Stokes equations in Ω:

$$\begin{cases} (\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot\mathbb{P}(\mathbf{v}, p) = \mathbf{f}, \qquad \nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \\ \mathbb{P}(\mathbf{v}, p)\mathbf{n}_\Gamma = 0 \quad \text{on } \Gamma, \\ \mathbb{P}(\mathbf{v}, p)\mathbf{n}_\Sigma\cdot\boldsymbol{\tau}_\Sigma = 0, \qquad \mathbf{v}\cdot\mathbf{n}_\Sigma = 0 \quad \text{on } \Sigma. \end{cases} \tag{1.1}$$

Here $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $\mathbf{v} = (v_1(x), v_2(x))$ is the velocity vector field, $p = p(x)$ is the pressure, $\mathbb{P}(\mathbf{v}, p) = -p\mathbb{I}_2 + 2\nu\mathbb{D}(\mathbf{v})$ is the stress tensor, $\mathbb{I}_2$ is an identity matrix of degree 2, $\mathbb{D}(\mathbf{v})$ is the velocity deformation tensor with the elements $D_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$ $(i, j = 1, 2)$, $\nu > 0$ is a constant coefficient of viscosity, $\mathbf{f} = (f_1(x), f_2(x))$

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is a given vector field of external forces, $\mathbf{n}_\Gamma$ is a unit outward normal vector to Γ, and $\mathbf{n}_\Sigma$ and $\boldsymbol{\tau}_\Sigma$ are a unit outward normal and a unit tangential vector to Σ such that $\mathbf{n}_\Sigma \times \boldsymbol{\tau}_\Sigma = 1$.

Problem (1.1) is related to the time-dependent problem with a free boundary describing the evolution of viscous incompressible fluid when the free surface has contact with the fixed boundary.

We study this problem (1.1) in weighted Sobolev spaces. Let $l = 0, 1, 2, \dots$ and $\mu \in \mathbb{R}$. By $\mathrm{H}^l_\mu(\Omega)$ we mean the space of functions $u(x)$, $x \in \Omega$, equipped with the norm

$$\|u\|^2_{\mathrm{H}^l_\mu(\Omega)} = \sum_{|\alpha| \le l} \int_\Omega \varrho_M^{2(\mu - l + |\alpha|)}(x) |D^\alpha u(x)|^2 \, \mathrm{d}x,$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $D^\alpha u = (\frac{\partial}{\partial x_1})^{\alpha_1} (\frac{\partial}{\partial x_2})^{\alpha_2} u$, $|\alpha| = \alpha_1 + \alpha_2$ and $\varrho_M(x)$ is a distance between $x \in \Omega$ and $M = \{x^{(1)}, x^{(2)}\}$.

The spaces of traces of functions in $\mathrm{H}^{l+1}_\mu(\Omega)$ on Γ and Σ are spaces $\mathrm{H}^{l+1/2}_\mu(\Gamma)$ and $\mathrm{H}^{l+1/2}_\mu(\Sigma)$, respectively. These spaces coincide with $\mathrm{H}^{l+1/2}_\mu(\Omega)$ whose norm is given by

$$\|u\|^2_{\mathrm{H}^{l+1/2}_\mu(\Omega)} = \|u\|^2_{\mathrm{H}^l_{\mu-1/2}(\Omega)} + \sum_{|\alpha| = l} \int_\Omega \varrho_M^{2\mu}(x) \, \mathrm{d}x \int_{|x-y| \le \varrho_M(x)/2} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^3} \, \mathrm{d}y.$$

2D vector valued functions are denoted by bold-faced letters such as $\mathbf{v} = (v_1, v_2)$. Similarly, we use the bold-faced letters to denote the function spaces of 2D vector valued functions.

Our main theorem is as follows.

Theorem 1.1. *Suppose that the non-negative integer l and the real number μ satisfy $0 < l + 1 - \mu < \lambda^*$, where the number λ^* is the smallest positive root to the equation $\sin 2\lambda\theta + \lambda \sin 2\theta = 0$ and $\mathbf{f} \in \mathbf{H}^l_\mu(\Omega)$ satisfies the some smallness condition (3.4) (see below), then there exists a unique solution $(\mathbf{v}, \nabla p) \in \mathbf{H}^{l+2}_\mu(\Omega) \times \mathrm{H}^l_\mu(\Omega)$ to problem (1.1), which satisfies the inequality*

$$\|\mathbf{v}\|_{\mathbf{H}^{l+2}_\mu(\Omega)} + \|\nabla p\|_{\mathrm{H}^l_\mu(\Omega)} \le c \|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)}.$$

The Proof of Theorem 1.1 depends on the investigation of the linearized problem and the contraction mapping principle. In studying the linearized problem, we follow [3,6,7].

2. Steady Stokes problem

In this section we consider the problem

$$\begin{cases} -\nabla \cdot \mathbb{P}(\mathbf{v}, p) = \mathbf{f}, \qquad \nabla \cdot \mathbf{v} = g \quad \text{in } \Omega, \\ \mathbb{P}(\mathbf{v}, p)\mathbf{n}_\Gamma = \mathbf{b} \quad \text{on } \Gamma, \\ \mathbb{P}(\mathbf{v}, p)\mathbf{n}_\Sigma \cdot \boldsymbol{\tau}_\Sigma = \mathbf{b}' \cdot \boldsymbol{\tau}_\Sigma = b_3, \qquad \mathbf{v} \cdot \mathbf{n}_\Sigma = b_4 \quad \text{on } \Sigma. \end{cases} \tag{2.1}$$

We prove

Theorem 2.1. *Let l, μ and λ^* be the same as those in Theorem 1.1. Then for any $\mathbf{h} = (\mathbf{f}, g, \mathbf{b}, b_3, b_4) \in Y \equiv \mathbf{H}^l_\mu(\Omega) \times \mathrm{H}^{l+1}_\mu(\Omega) \times \mathbf{H}^{l+1/2}_\mu(\Gamma) \times \mathrm{H}^{l+1/2}_\mu(\Sigma) \times \mathrm{H}^{l+3/2}_\mu(\Sigma)$ satisfying the condition*

$$\int_\Omega \mathbf{f} \, \mathrm{d}x = \int_\Gamma \mathbf{b} \, \mathrm{d}S + \int_\Sigma \mathbf{b}' \, \mathrm{d}S,$$

problem (2.1) has a unique, up to the rigid motion, solution $(\mathbf{v}, \nabla p) \in X \equiv \mathbf{H}^{l+2}_\mu(\Omega) \times \mathrm{H}^l_\mu(\Omega)$. Moreover, this solution satisfies the inequality

$$\begin{aligned} \|(\mathbf{v}, \nabla p)\|_X &\equiv \|\mathbf{v}\|_{\mathbf{H}^{l+2}_\mu(\Omega)} + \|\nabla p\|_{\mathrm{H}^l_\mu(\Omega)} \\ &\le c_1 \left(\|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)} + \|g\|_{\mathrm{H}^{l+1}_\mu(\Omega)} + \|\mathbf{b}\|_{\mathbf{H}^{l+1}_\mu(\Gamma)} + \|b_3\|_{\mathrm{H}^{l+1/2}_\mu(\Sigma)} + \|b_4\|_{\mathrm{H}^{l+3/2}_\mu(\Sigma)} \right) \\ &\equiv c_1 \|\mathbf{h}\|_Y. \end{aligned}$$

We first discuss the uniqueness of solution.

Lemma 2.1. *The solution* $(\mathbf{v}, \nabla p) \in X$ *to problem* (2.1) *is unique up to the rigid motion.*

Proof. Let $(\mathbf{v}, \nabla p)$ be a solution of (2.1) with $\mathbf{f} = g = \mathbf{b} = b_3 = b_4 = 0$. Then we have by integration by parts

$$\begin{aligned}
0 &= -\int_{\Omega} (\nabla \cdot \mathbb{P}(\mathbf{v}, p)) \cdot \mathbf{v}\, \mathrm{d}x \\
&= -\int_{\Gamma \cup \Sigma} \mathbb{P}(\mathbf{v}, p)\mathbf{n} \cdot \mathbf{v}\, \mathrm{d}S + 2\nu \int_{\Omega} \mathbb{D}(\mathbf{v}) : \mathbb{D}(\mathbf{v})\, \mathrm{d}x \\
&= 2\nu \int_{\Omega} \mathbb{D}(\mathbf{v}) : \mathbb{D}(\mathbf{v})\, \mathrm{d}x.
\end{aligned}$$

From this it follows that $\mathbb{D}(\mathbf{v}) = 0$, and hence $\nabla p = 0$. $\square$

The solvability of (2.1) will be proved by the method of regularizer, for which it is necessary to introduce two systems of coverings $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ of $\bar{\Omega}$.

For an arbitrary small positive number λ, coverings $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ have the following properties:

1. $\omega^{(k)} \subset \Omega^{(k)} \subset \bar{\Omega}$, $\bigcup_{k\in\mathbb{N}} \omega^{(k)} = \bigcup_{k\in\mathbb{N}} \Omega^{(k)} = \bar{\Omega}$;
2. For any $x \in \bar{\Omega}$, there exists $\omega^{(k)}$ such that $x \in \omega^{(k)}$ and $\mathrm{dist}(x, \bar{\Omega} \setminus \omega^{(k)}) \geq \delta_1\lambda$ for some $\delta_1 > 0$;
3. For any $\lambda > 0$, there exists a positive integer N_0 independent of λ such that $\bigcap_{k=1}^{N_0+1} \Omega^{(k)} = \phi$;

4(i). For $k = 1, 2$ we set $\Omega^{(k)} = \Omega \cap B(x^{(k)}; \lambda) \subset \Omega \cap U^{(k)}$, $\omega^{(k)} = \Omega \cap B(x^{(k)}; \lambda/2)$.
We decompose $\{k \in \mathbb{N} | k \geq 3\}$ into three groups:

4(ii). We denote by $k \in \mathbb{N}_1$ if $\Omega^{(k)} \cap (\Gamma^c \cup \Sigma^c) = \phi$, $x^{(i)} \notin \bar{\Omega}^{(k)}$ $(i = 1, 2)$, where $\Gamma^c = \Gamma \cap \bar{\Omega}^c$, $\Sigma^c = \Sigma \cap \bar{\Omega}^c$ with $\Omega^c = \Omega \setminus (\Omega^{(1)} \cup \Omega^{(2)})$.

4(iii). We denote by $k \in \mathbb{N}_2$ if $\omega^{(k)} \cap \Gamma^c \neq \phi$, $\Omega^{(k)} \cup \Sigma^c = \phi$, $x^{(i)} \notin \bar{\Omega}^{(k)}$ $(i = 1, 2)$.

4(iv). Finally we denote by $k \in \mathbb{N}_3$ if $\omega^{(k)} \cap \Sigma^c \neq \phi$, $\Omega^{(k)} \cup \Gamma^c = \phi$, $x^{(i)} \notin \bar{\Omega}^{(k)}$ $(i = 1, 2)$.

Such coverings $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}_{k\in\{1,2\}\cup\mathbb{N}_1\cup\mathbb{N}_2\cup\mathbb{N}_3}$ are constructed in a standard manner [6,2].

It is well-known that passage to a local coordinate system $\{y\}$ from the original one $\{x\}$, given by the relation $y = \mathbb{L}^{(k)}(x - x^{(k)})$ near $x^{(k)} \in \Gamma \cup \Sigma$ with an orthogonal matrix $\mathbb{L}^{(k)}$ accompanied by a corresponding linear transformation of a vector field $\mathbf{v}' = \mathbb{L}^{(k)}\mathbf{v}$, keeps problem (2.1) invariant. Therefore without loss of generality we may assume that $x^{(k)} = 0$ and $\{x\}$ is a local coordinate system.

Let $\{\zeta^{(k)}(x)\}$ be a smooth partition of unity subordinated to the coverings $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ of $\bar{\Omega}$ introduced above, which satisfies the inequalities

$$|D^{\alpha}\zeta^{(k)}(x)| \leq c\lambda^{-|\alpha|}, \quad \forall\alpha \text{ (multi-index)} \tag{2.2}$$

with some constant c independent of λ and k. Let $\{\eta^{(k)}(x)\}$ be a function defined by $\eta^{(k)}(x) = \zeta^{(k)}(x)/\sum_j \zeta^{(j)}(x)^2$. It is obvious that $\eta^{(k)}(x)$ also satisfies the inequality of type (2.2) and $\sum_k \zeta^{(k)}(x)\eta^{(k)}(x) = 1$. For $k = 1, 2$, we denote by Π_z^x the transformation from $\Omega^{(k)}$ to $d_{\theta^{(k)}}$, while for $k \in \mathbb{N}_2$ and $k \in \mathbb{N}_3$ we use the same symbol Π_z^x to denote the coordinate transformation from $\{x\}$ to $\{z\}$ which rectifies the boundary Γ and Σ, respectively. Then we introduce the operator $\mathcal{R}$ defined by

$$\mathcal{R}\mathbf{h} = \sum_k \eta^{(k)}(x)(\mathbf{v}^{(k)}, \nabla p^{(k)})(x) = \sum_{k\in\mathbb{N}_1} \eta^{(k)}(x)(\mathbf{v}^{(k)}, \nabla p^{(k)})(x) + \sum_{k\in\{1,2\}\cup\mathbb{N}_2\cup\mathbb{N}_3} \eta^{(k)}(x)\Pi_x^z(\bar{\mathbf{v}}^{(k)}, \nabla\bar{p}^{(k)})(z),$$

where Π_x^z is the inverse transformation of Π_z^x, $(\mathbf{v}^{(k)}, \nabla p^{(k)})(x)(k \in \mathbb{N}_1)$ and $(\bar{\mathbf{v}}^{(k)}, \nabla\bar{p}^{(k)})(z)(k \in \{1, 2\} \cup \mathbb{N}_2 \cup \mathbb{N}_3)$ are the solution of following problems, respectively.

(i) For $k = 1, 2$ (problem in a plane angle)

$$\begin{cases}
-\nu\Delta\bar{\mathbf{v}}^{(k)} + \nabla\bar{p}^{(k)} = \Pi_z^x\zeta^{(k)}\mathbf{f}, \qquad \nabla\cdot\bar{\mathbf{v}}^{(k)} = \Pi_z^x\zeta^{(k)}g \quad \text{in } d_{\theta^{(k)}}, \\
\mathbb{P}(\bar{\mathbf{v}}^{(k)}, \bar{p}^{(k)})\mathbf{n} = \Pi_z^x\,\zeta^{(k)}\mathbf{b} \quad \text{on } \gamma_{\theta^{(k)}}, \\
2\nu\mathbb{D}(\bar{\mathbf{v}}^{(k)})\mathbf{n}_0\cdot\boldsymbol{\tau}_0 = \Pi_z^x\zeta^{(k)}b_3, \qquad \bar{\mathbf{v}}^{(k)}\cdot\mathbf{n}_0 = \Pi_z^x\zeta^{(k)}b_4 \quad \text{on } \gamma_0;
\end{cases} \tag{2.3}$$

(ii) For $k \in \mathbb{N}_1$ (problem in a whole space)

$$-\nu\Delta\mathbf{v}^{(k)} + \nabla p^{(k)} = \zeta^{(k)}\mathbf{f}, \qquad \nabla\cdot\mathbf{v}^{(k)} = \zeta^{(k)}g \quad \text{in } \mathbb{R}^{2(k)} \equiv \Pi_z^x\Omega^{(k)}; \tag{2.4}$$

(iii) For $k \in \mathbb{N}_2$ (problem with Neumann condition)

$$\begin{cases} -\nu\Delta\bar{\mathbf{v}}^{(k)} + \nabla\bar{p}^{(k)} = \Pi_z^x\zeta^{(k)}\mathbf{f}, \qquad \nabla\cdot\bar{\mathbf{v}}^{(k)} = \Pi_z^x\zeta^{(k)}g \quad \text{in } \mathbb{R}_+^{2(k)}, \\ \mathbb{P}(\bar{\mathbf{v}}^{(k)}, \bar{p}^{(k)})\mathbf{n}_0 = \Pi_z^x\zeta^{(k)}\mathbf{b} \quad \text{on } z_2 = 0, \end{cases} \tag{2.5}$$

where $\mathbb{R}_+^{2(k)} \equiv \{z = (z_1, z_2) \in \mathbb{R}^{2(k)} \mid z_2 > 0\}$;

(iv) For $k \in \mathbb{N}_3$ (problem with perfect slip condition)

$$\begin{cases} -\nu\Delta\bar{\mathbf{v}}^{(k)} + \nabla\bar{p}^{(k)} = \Pi_z^x\zeta^{(k)}\mathbf{f}, \qquad \nabla\cdot\bar{\mathbf{v}}^{(k)} = \Pi_z^x\zeta^{(k)}g \quad \text{in } \mathbb{R}_+^{2(k)}, \\ 2\nu\mathbb{D}(\bar{\mathbf{v}}^{(k)})\mathbf{n}_0\cdot\boldsymbol{\tau}_0 = \Pi_z^x\zeta^{(k)}b_3, \qquad \bar{\mathbf{v}}^{(k)}\cdot\mathbf{n}_0 = \Pi_z^x\zeta^{(k)}b_4 \quad \text{on } z_2 = 0, \end{cases} \tag{2.6}$$

where $\mathbf{n}_0 = (0, -1)^{\mathrm{T}}$ and $\boldsymbol{\tau}_0 = (1, 0)^{\mathrm{T}}$.

Problem (2.3) was studied in [1]:

Lemma 2.2. *Let l, μ and λ^* be the same as those in Theorem 1.1. Then for any $\mathbf{f} = (f_1, f_2) \in \mathbf{H}_\mu^l(d_\theta)$, $g \in \mathrm{H}_\mu^{l+1}(d_\theta)$, $\mathbf{b} = (b_1, b_2) \in \mathbf{H}_\mu^{l+1/2}(\mathbb{R}_+)$, $\mathbf{b}'\cdot\boldsymbol{\tau}_0 = b_3 \in \mathrm{H}_\mu^{l+1/2}(\mathbb{R}_+)$, $b_4 \in \mathrm{H}_\mu^{l+3/2}(\mathbb{R}_+)$ satisfying the condition*

$$\int_{d_\theta}(\mathbf{f} - \nu\nabla g)\,\mathrm{d}x = \int_{\gamma_\theta}\mathbf{b}\,\mathrm{d}x_1 + \int_{\gamma_0}\mathbf{b}'\,\mathrm{d}r,$$

then problem

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, \qquad \nabla\cdot\mathbf{v} = g \quad \text{in } d_\theta, \\ \mathbb{P}(\mathbf{v}, p)\mathbf{n} = \mathbf{b} \quad \text{on } \gamma_\theta, \\ 2\nu\mathbb{D}(\mathbf{v})\mathbf{n}_0\cdot\boldsymbol{\tau}_0 = b_3, \qquad \mathbf{v}\cdot\mathbf{n}_0 = b_4 \quad \text{on } \gamma_0 \end{cases}$$

has a unique solution $\mathbf{v} \in \mathbf{H}_\mu^{l+2}(d_\theta)$, $\nabla p \in \mathrm{H}_\mu^l(d_\theta)$. This solution satisfies the inequality

$$\|\mathbf{v}\|_{\mathbf{H}_\mu^{l+2}(d_\theta)} + \|\nabla p\|_{\mathrm{H}_\mu^l(d_\theta)} \le c\left(\|\mathbf{f}\|_{\mathbf{H}_\mu^l(d_\theta)} + \|g\|_{\mathrm{H}_\mu^{l+1}(d_\theta)} + \|\mathbf{b}\|_{\mathbf{H}_\mu^{l+1/2}(\mathbb{R}_+)} + \|b_3\|_{\mathrm{H}_\mu^{l+1/2}(\mathbb{R}_+)} + \|b_4\|_{\mathrm{H}_\mu^{l+3/2}(\mathbb{R}_+)}\right),$$

where c is a positive constant independent of $\mathbf{f}$, g, $\mathbf{b}$, b_3 *and* b_4.

On the other hand problems (2.4)–(2.6) were discussed in [4,6,2].

Defining the operator $\mathcal{A}$ by

$$\mathcal{A}(\mathbf{v}, p) = (-\Delta\mathbf{v} + \nabla p, \nabla\cdot\mathbf{v}, \mathbb{P}(\mathbf{v}, p)\mathbf{n}|_\Gamma, \mathbb{P}(\mathbf{v}, p)\mathbf{n}\cdot\boldsymbol{\tau}|_\Sigma, \mathbf{v}\cdot\mathbf{n}|_\Sigma),$$

after some lengthy calculations we have

$$\mathcal{A}\mathcal{R}\mathbf{h} = \mathbf{h} + \mathcal{M}\mathbf{h} = \mathbf{h} + \mathcal{T}\mathbf{h} + \mathcal{K}\mathbf{h}, \tag{2.7}$$

where

$$\begin{aligned} &\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5), \\ &\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, 0), \\ &\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5), \\ &\mathcal{M}_1\mathbf{h} = \sum_k\left(-\nu(\Delta(\eta^{(k)}\mathbf{v}^{(k)}) - \eta^{(k)}\Delta\mathbf{v}^{(k)}) + \nabla(\eta^{(k)}p^{(k)}) - \eta^{(k)}\nabla p^{(k)}\right) \\ &\qquad + \sum_{k\in\{1,2\}\cup\mathbb{N}_2\cup\mathbb{N}_3}\eta^{(k)}\Pi_x^z\left(-\nu(\Delta^{(k)} - \Delta)\bar{\mathbf{v}}^{(k)} + (\nabla^{(k)} - \nabla)\bar{p}^{(k)}\right) \equiv \mathcal{T}_1\mathbf{h} + \mathcal{K}_1\mathbf{h}, \\ &\mathcal{M}_2\mathbf{h} = \sum_k\left(\nabla\cdot(\eta^{(k)}\mathbf{v}^{(k)}) - \eta^{(k)}\nabla\cdot\mathbf{v}^{(k)}\right) + \sum_{k\in\{1,2\}\cup\mathbb{N}_2\cup\mathbb{N}_3}\eta^{(k)}\Pi_x^z(\nabla^{(k)} - \nabla)\cdot\bar{\mathbf{v}}^{(k)} \equiv \mathcal{T}_2\mathbf{h} + \mathcal{K}_2\mathbf{h}, \end{aligned}$$

$$\begin{aligned}
\mathcal{M}_3\mathbf{h} &= \sum_{k\in\{1,2\}\cup\mathbb{N}_2} \left(\mathbb{P}(\eta^{(k)}\mathbf{v}^{(k)}, \eta^{(k)}p^{(k)})\mathbf{n} - \eta^{(k)}\mathbb{P}(\mathbf{v}^{(k)}, p^{(k)})\mathbf{n}\right) \\
&\quad + \sum_{k\in\{1,2\}\cup\mathbb{N}_2} \eta^{(k)}\Pi_x^z \left(\mathbb{P}^{(k)}(\bar{\mathbf{v}}^{(k)}, \bar{p}^{(k)})\mathbf{n} - \mathbb{P}(\bar{\mathbf{v}}^{(k)}, \bar{p}^{(k)})\mathbf{n}_0\right) \equiv \mathcal{T}_3\mathbf{h} + \mathcal{K}_3\mathbf{h}, \\
\mathcal{M}_4\mathbf{h} &= 2\nu \sum_{k\in\{1,2\}\cup\mathbb{N}_3} \left(\mathbb{D}(\eta^{(k)}\mathbf{v}^{(k)})\mathbf{n}\cdot\boldsymbol{\tau} - \eta^{(k)}\mathbb{D}(\mathbf{v}^{(k)})\mathbf{n}\cdot\boldsymbol{\tau}\right) \\
&\quad + 2\nu \sum_{k\in\{1,2\}\cup\mathbb{N}_3} \eta^{(k)}\Pi_x^z \left(\mathbb{D}^{(k)}(\bar{\mathbf{v}}^{(k)})\mathbf{n}\cdot\boldsymbol{\tau} - \mathbb{D}(\bar{\mathbf{v}}^{(k)})\mathbf{n}_0\cdot\boldsymbol{\tau}_0\right) \equiv \mathcal{T}_4\mathbf{h} + \mathcal{K}_4\mathbf{h}, \\
\mathcal{M}_5\mathbf{h} &= \sum_{k\in\{1,2\}\cup\mathbb{N}_3} \eta^{(k)}\Pi_x^z\bar{\mathbf{v}}^{(k)}\cdot(\mathbf{n}-\mathbf{n}_0) \equiv \mathcal{K}_5\mathbf{h}, \\
\nabla^{(k)} &= \Pi_z^x\nabla_x, \qquad \Delta^{(k)} = \nabla^{(k)}\cdot\nabla^{(k)}, \qquad \mathbb{D}^{(k)} = \Pi_z^x\mathbb{D}, \qquad \mathbb{P}^{(k)} = \Pi_z^x\mathbb{P}.
\end{aligned}$$

We can show that $\mathcal{K}$ is a contraction operator on Y for small λ by introducing the norm dependent on parameter λ (see [6,2]), while $\mathcal{T}$ is a compact operator on Y for each λ, since the imbedding operator from $\mathrm{H}^l_\mu(\Omega)$ into $\mathrm{H}^l_\mu(\Omega)$ is compact for a bounded domain Ω [5, p. 98]. Therefore, from (2.7) it implies the existence of the right regularizer.

Now let us consider $\mathcal{RA}(\mathbf{v}, \nabla p)$. Similar calculations as above yield

$$\mathcal{RA}(\mathbf{v}, \nabla p) = (\mathbf{v}, \nabla p) + \mathcal{S}(\mathbf{v}, \nabla p) + \mathcal{Q}(\mathbf{v}, \nabla p), \tag{2.8}$$

where $(\mathbf{v}, \nabla p) \in X$, $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, 0)$, $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5)$,

$$\begin{aligned}
\mathcal{S}_1(\mathbf{v}, \nabla p) &= \sum_k \eta^{(k)}\Pi_x^z\mathcal{R}^{(k)}\Pi_z^x \left(-\nu(\zeta^{(k)}\Delta\mathbf{v} - \Delta(\zeta^{(k)}\mathbf{v})) + \zeta^{(k)}\nabla p - \nabla(\zeta^{(k)}p)\right), \\
\mathcal{Q}_1(\mathbf{v}, \nabla p) &= \sum_{k\in\{1,2\}\cup\mathbb{N}_2\cup\mathbb{N}_3} \eta^{(k)}\Pi_x^z\mathcal{R}^{(k)} \left(-\nu(\Delta^{(k)} - \Delta)\bar{\mathbf{v}}^{(k)} + (\nabla^{(k)} - \nabla)\bar{p}^{(k)}\right), \\
\mathcal{S}_2(\mathbf{v}, \nabla p) &= \sum_k \eta^{(k)}\Pi_x^z\mathcal{R}^{(k)}\Pi_z^x \left(\zeta^{(k)}\nabla\cdot\mathbf{v} - \nabla\cdot(\zeta^{(k)}\mathbf{v})\right), \\
\mathcal{Q}_2(\mathbf{v}, \nabla p) &= \sum_{k\in\{1,2\}\cup\mathbb{N}_2\cup\mathbb{N}_3} \eta^{(k)}\Pi_x^z\mathcal{R}^{(k)} \left(\nabla^{(k)} - \nabla\right)\cdot\bar{\mathbf{v}}^{(k)}, \\
\mathcal{S}_3(\mathbf{v}, \nabla p) &= \sum_{k\in\{1,2\}\cup\mathbb{N}_2} \eta^{(k)}\Pi_x^z\mathcal{R}^{(k)}\Pi_z^x \left(\zeta^{(k)}\mathbb{P}(\mathbf{v}, p)\mathbf{n} - \mathbb{P}(\zeta^{(k)}\mathbf{v}, p)\mathbf{n}\right), \\
\mathcal{Q}_3(\mathbf{v}, \nabla p) &= \sum_{k\in\{1,2\}\cup\mathbb{N}_2} \eta^{(k)}\Pi_x^z\mathcal{R}^{(k)} \left(\mathbb{D}^{(k)}(\bar{\mathbf{v}}^{(k)}, \bar{p}^{(k)})\mathbf{n} - \mathbb{D}(\bar{\mathbf{v}}^{(k)}, \bar{p}^{(k)})\mathbf{n}_0\right), \\
\mathcal{S}_4(\mathbf{v}, \nabla p) &= 2\nu \sum_{k\in\{1,2\}\cup\mathbb{N}_3} \eta^{(k)}\Pi_x^z\mathcal{R}^{(k)}\Pi_z^x \left(\zeta^{(k)}\mathbb{D}(\mathbf{v})\mathbf{n} - \mathbb{D}(\zeta^{(k)}\mathbf{v})\mathbf{n}\right)\cdot\boldsymbol{\tau}, \\
\mathcal{Q}_4(\mathbf{v}, \nabla p) &= 2\nu \sum_{k\in\{1,2\}\cup\mathbb{N}_3} \eta^{(k)}\Pi_x^z\mathcal{R}^{(k)} \left(\mathbb{D}^{(k)}(\bar{\mathbf{v}}^{(k)})\mathbf{n}\cdot\boldsymbol{\tau} - \mathbb{D}(\bar{\mathbf{v}}^{(k)})\mathbf{n}_0\cdot\boldsymbol{\tau}_0\right), \\
\mathcal{Q}_5(\mathbf{v}, \nabla p) &= \sum_{k\in\{1,2\}\cup\mathbb{N}_3} \eta^{(k)}\Pi_x^z\mathcal{R}^{(k)}\Pi_z^x\zeta^{(k)}\mathbf{v}\cdot(\mathbf{n}-\mathbf{n}_0),
\end{aligned}$$

where $\mathcal{R}^{(k)}$ are the solution operators of the problems in $\Omega^{(k)}(k \in \mathbb{N}_1)$ or in $\Pi_z^x\Omega^{(k)}(k \in \{1,2\}\cup\mathbb{N}_2\cup\mathbb{N}_3)$. By exactly the same way as $\mathcal{K}$ and $\mathcal{T}$, one can show that $\mathcal{Q}$ is a contraction operator on X and that $\mathcal{S}$ is a compact operator on X, from which together with (2.8) it follows the existence of the left regularizer. By combining these and the uniqueness of a solution from Lemma 2.1, Theorem 2.1 is proved. □

3. Steady Navier–Stokes problem

We first note the following:

Lemma 3.1. *If $u \in \mathrm{H}^{l+2}_\mu(d_\theta)$, $v \in \mathrm{H}^l_\mu(d_\theta)$ and $u = v = 0$ for $|x| \geq R > 0$, then*

$$\|uv\|_{\mathrm{H}^l_\mu(d_\theta)} \leq c_2 \|u\|_{\mathrm{H}^{l+2}_\mu(d_\theta)} \|v\|_{\mathrm{H}^l_\mu(d_\theta)}, \tag{3.1}$$

where constant c_2 depends on R.

Proof.

$$\begin{aligned}\|uv\|^2_{\mathrm{H}^l_\mu(d_\theta)} &= \sum_{|\alpha|\leq l} \int_{d_\theta} |x|^{2(\mu-l+|\alpha|)}(x)|D^\alpha(uv)(x)|^2\,\mathrm{d}x \\ &\leq \sum_{|\alpha|\leq l}\sum_{|\beta|\leq|\alpha|} C_{\alpha\beta}\int_{d_\theta} |x|^{2(\mu-l+|\alpha-\beta|)}|D^{\alpha-\beta}v(x)|^2\cdot|x|^{2|\beta|}|D^\beta u(x)|^2\,\mathrm{d}x.\end{aligned}$$

From inequality (7.24) in [5, p. 88], we find

$$|x|^{|\beta|}|D^\beta u(x)| \leq c|x|^{(l+2)-\mu-1}\|u\|^2_{\mathrm{H}^{l+2}_\mu(d_\theta)},$$

which easily yields (3.1). □

We can obtain the inequality similar to (3.1) for Ω by a standard argument.

Now we solve problem (1.1) by the method of successive approximations. Let

$$(\mathbf{v}^{(0)}, \nabla p^{(0)}) = (0, 0)$$

and for a given

$$(\mathbf{v}^{(m)}, \nabla p^{(m)}) \in X_\mathbf{f} \equiv \left\{(\mathbf{v}, \nabla p) \in X \mid \|(\mathbf{v}, \nabla p)\|_X \leq 2c_1\|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)}\right\}$$

$(m = 0, 1, 2, \ldots)$ let $(\mathbf{v}^{(m+1)}, \nabla p^{(m+1)})$ be a solution to the linear problem

$$\begin{cases} -\nu\Delta\mathbf{v}^{(m+1)} + \nabla p^{(m+1)} = \mathbf{f} - (\mathbf{v}^{(m)}\cdot\nabla)\mathbf{v}^{(m)}, \\ \nabla\cdot\mathbf{v}^{(m+1)} = 0 \quad \text{in } \Omega, \\ \mathbb{P}(\mathbf{v}^{(m+1)}, p^{(m+1)})\mathbf{n} = 0 \quad \text{on } \Gamma, \\ \mathbb{P}(\mathbf{v}^{(m+1)}, p^{(m+1)})\mathbf{n}\cdot\boldsymbol{\tau} = 0, \qquad \mathbf{v}^{(m+1)}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma. \end{cases} \tag{3.2}$$

By Theorem 2.1 problem (3.2) has a unique, up to the rigid motion, solution $(\mathbf{v}^{(m+1)}, \nabla p^{(m+1)}) \in X$ satisfying

$$\begin{aligned}\|(\mathbf{v}^{(m+1)}, \nabla p^{(m+1)})\|_X &\leq c_1(\|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)} + \|(\mathbf{v}^{(m)}\cdot\nabla)\mathbf{v}^{(m)}\|_{\mathbf{H}^l_\mu(\Omega)}) \\ &\leq c_1(\|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)} + c_2\|\mathbf{v}^{(m)}\|_{\mathbf{H}^{l+2}_\mu(\Omega)}\|\mathbf{v}^{(m)}\|_{\mathbf{H}^{l+1}_\mu(\Omega)}) \\ &\leq c_1(\|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)} + c_2(2c_1\|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)})^2) \\ &\leq c_1(1 + 4c_1^2c_2\|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)})\|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)}.\end{aligned} \tag{3.3}$$

Hence we find $(\mathbf{v}^{(m+1)}, \nabla p^{(m+1)}) \in X_\mathbf{f}$ provided

$$4c_1^2c_2\|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)} < 1. \tag{3.4}$$

Now let us prove the convergence of the sequence $\{(\mathbf{v}^{(m)}, \nabla p^{(m)})\}_{m=0,1,2,\ldots}$. Subtracting from (3.2) the similar equations for $(\mathbf{v}^{(m)}, \nabla p^{(m)})$ and setting $(\mathbf{V}^{(m+1)}, \nabla P^{(m+1)}) = (\mathbf{v}^{(m+1)} - \mathbf{v}^{(m)}, \nabla p^{(m+1)} - \nabla p^{(m)})$, we obtain

$$\begin{cases} -\nu\Delta\mathbf{V}^{(m+1)} + \nabla P^{(m+1)} = -(\mathbf{v}^{(m)}\cdot\nabla)\mathbf{V}^{(m)} - (\mathbf{V}^{(m)}\cdot\nabla)\mathbf{v}^{(m-1)}, \\ \nabla\cdot\mathbf{V}^{(m+1)} = 0 \quad \text{in } \Omega, \\ \mathbb{P}(\mathbf{V}^{(m+1)}, P^{(m+1)})\mathbf{n} = 0 \quad \text{on } \Gamma, \\ \mathbb{P}(\mathbf{V}^{(m+1)}, P^{(m+1)})\mathbf{n}\cdot\boldsymbol{\tau} = 0, \qquad \mathbf{V}^{(m+1)}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma. \end{cases} \tag{3.5}$$

By virtue of Theorem 2.1, there exists a solution $(\mathbf{V}^{(m+1)}, \nabla P^{(m+1)}) \in X$ of (3.5), which satisfies

$$\begin{aligned}
\|(\mathbf{V}^{(m+1)}, \nabla P^{(m+1)})\|_X &\leq c_1 \left(\|(\mathbf{v}^{(m)} \cdot \nabla)\mathbf{V}^{(m)}\|_{\mathbf{H}^l_\mu(\Omega)} + \|(\mathbf{V}^{(m)} \cdot \nabla)\mathbf{v}^{(m-1)}\|_{\mathbf{H}^l_\mu(\Omega)} \right) \\
&\leq c_1 c_2 \left(\|\mathbf{v}^{(m)}\|_{\mathbf{H}^{l+2}_\mu(\Omega)} \|\nabla \mathbf{V}^{(m)}\|_{\mathbf{H}^l_\mu(\Omega)} + \|\mathbf{V}^{(m)}\|_{\mathbf{H}^{l+2}_\mu(\Omega)} \|\nabla \mathbf{v}^{(m-1)}\|_{\mathbf{H}^l_\mu(\Omega)} \right) \\
&\leq 4c_1^2 c_2 \|\mathbf{f}\|_{\mathbf{H}^l_\mu(\Omega)} \|(\mathbf{V}^{(m)}, \nabla P^{(m)})\|_X. \qquad (3.6)
\end{aligned}$$

Therefore under the condition (3.4) we see that the sequence $(\mathbf{v}^{(m)}, \nabla p^{(m)})$ converges to some $(\mathbf{v}, \nabla p) \in X_{\mathbf{f}}$ as $m \to \infty$, which is our desired solution to problem (1.1).

The uniqueness of the solution easily follows from the estimate similar to inequality (3.6). □

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